Note: all statements require proofs. You can make references to standard theorems from the course; however, you must state the relevant part of the theorem in your own words, unless it is a well-known named theorem. For example, "by the monotone convergence theorem," or, "we showed in lecture that the integrals of an increasing sequence of positive functions converge to the integral of their limit," are good references but, "by a convergence theorem the integrals converge," is **not** a good reference.

1. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Suppose the quantity

$$\alpha := \inf\{\mu(E) \colon E \in \mathcal{M} \text{ with } \mu(E) > 0\}$$

is strictly positive. Show that there exists an **atom** $A \in \mathcal{M}$; that is, $\mu(A) > 0$ and $\mu(B) \in \{0, \mu(A)\}$ for all measurable $B \subset A$.

[**Hint:** if $E \in \mathcal{M}$ is not an atom but has $\mu(E) > 0$, show there exists $F \subset E$ with $\mu(F) \in (0, \frac{1}{2}\mu(E)]$.]

2. $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. We say an $\mathcal{M}$-measurable function $f \colon X \to \mathbb{C}$ is μ**-finitely supported** if there exists $E \in \mathcal{M}$ with $\mu(E) < \infty$ and such that $f|_{X \setminus E} \equiv 0$. Show that the μ-finitely supported functions are dense in $L^1(X, \mu)$.

3. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and suppose $(f_n)_{n \in \mathbb{N}}$ is a sequence of $\mathcal{M}$-measurable functions converging μ-almost everywhere to some $f \in L^\infty(X, \mu)$. Show that there exists a decreasing sequence $X \supset E_1 \supset E_2 \supset \cdots$ with $\mu(E_k) \to 0$ and a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ so that

$$\lim_{k \to \infty} \|1_{X \setminus E_k}(f - f_{n_k})\|_\infty = 0.$$

4. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Show that for $1 \leq p < q \leq \infty$, one has $L^q(X, \mu) \subset L^p(X, \mu)$.

5. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and let $1 < p, q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Suppose ν is a σ-finite measure on $(X, \mathcal{M})$ satisfying:

   (i) $L^p(X, \mu) \subset L^1(X, \nu)$; and

   (ii) for any sequence $(f_n)_{n \in \mathbb{N}} \subset L^p(X, \mu)$ with $\|f_n\|_p \to 0$ one has

$$\lim_{n \to \infty} \int_X f_n \, d\nu = 0.$$

Show that $\nu \ll \mu$ and $\frac{d\nu}{d\mu} \in L^q(X, \mu)$.